Guidance Laws for Aerodynamically Controlled Re-entry Vehicles

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Linear-quadratic regulator theory has many attractive features which suggest its use in the synthesis of guidance laws. This paper investigates the relative controllability of the dynamic modes of a linearized model of an aerodynamically controlled re-entry vehicle and uses the results of this investigation to aid in the selection of the state weighting matrix in the performance index.

Nomenclature

a	= accelaration autopilot corner frequency, 1/sec
A	= vehicle acceleration, fps^2
A_c	= commanded vehicle acceleration, fps^2
b	= bank angle autopilot corner frequency, 1/sec
C	= controllability matrix
E	= measure of control energy
F	= perturbation system matrix
G	= perturbation control matrix
J	= performance index
M	= elements of time varying controllability matrix
$\mathfrak{N}$	= null space
P	= weighting matrix and solution to matrix Riccati equation
Q	= weighting matrix on state error
R	= weighting matrix on control error
t	= time, sec
T	= coordinate transformation matrix
u	= perturbation control vector
V	= velocity magnitude, fps
W	= integral associated with controllability
x	= perturbation state vector, target coordinates
X	= downrange distanct, ft
Y	= crossrange distance, ft
z	= perturbation state vector, "wind" coordinates
Z	= negative altitude, ft
γ	= flight-path angle, rad
Δ	= small time perturbation, sec
λ	= controllability factor
μ	= eigenvector
υ	= eigenvalue
ρ	= dummy derivative variable, sec
σ	= normalized eigenvector
τ	= dummy time variable, sec
ϕ	= bank angle, rad
ϕ_c	= commanded bank angle, rad
Φ	= transition matrix
ψ	= azimuth angle, rad

Superscripts

$(\dot{\ })$	= time derivative
$(\ ')$	= transpose
$(\bar{\ })$	= unit vector
$(\tilde{\ })$	= normalized control
$(\ ^*)$	= optimal

Subscripts

f	= final
m	= maximum
o	= initial
p	= perturbation

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Introduction

SYNTHESIS of a satisfactory guidance law for a re-entry vehicle involves allocating the available control effort in such a way that the vehicle follows a prescribed trajectory and impacts the ground near a specific point. When the dynamic structure of the vehicle is described by a set of ordinary differential equations, it is usually not difficult to derive an open-loop control which will cause the vehicle trajectory to satisfy the performance specifications. Unfortunately, open-loop guidance tends to be very sensitive to external disturbances and to vehicle parameter changes that cannot be predicted a priori. In this application these anomalous effects are of such size that satisfactory performance is unattainable with open-loop guidance.

To produce guidance laws that will continue to meet the performance specifications in the presence of disturbances, feedback is required. Simplification in the guidance structure is achieved if the vehicle trajectory and control variables are separated into two parts, a nominal part and a perturbed part. The nominal variables are selected to meet all performance specifications and represent ideal system behavior. The nominal controller is open-loop; i.e., the nominal actuating signal is a function of time alone. The perturbed variables are minimized through the use of a feedback controller. The dynamic equation of the perturbed variables is usually given by the linear terms in a Taylor's series expansion of the nonlinear dynamic representation of the vehicle. The complete guidance law is then the sum of the two constituent parts.

Although determination of the nominal control is typically straightforward, the feedback component of the guidance law is more difficult to derive. Uniform pointwise minimization of the deviation from nominal is impossible. A quadratic weighting of perturbed trajectory and control variables is often used as an index of performance. This permits an algorithmic approach to controller synthesis with the assurance that certain desirable technical attributes of system performance will ensue.[1] The resulting control rule is parameterized by the weightings used in the performance index; thus, the judicious choice of these weightings becomes of primary importance. A commonly used rule-of-thumb is to weight deviations from nominal with a multiplier inversely proportional to the square of the allowable deviation.[2] Such weightings are often unsatisfactory and it is then suggested that the weighting matrices be iteratively modified on either an ad hoc basis or algorithmically if the system specifications are phrased appropriately.[3-6]

The dynamic equations of an aerodynamically controlled re-entry vehicle possess certain technical pecularities which cause system response to be quite sensitive to the parameters of the performance index. While the perturbation equations are of the generic class to which linear-quadratic theory is usually applied, unacceptable guidance laws may result from commonly used criterion functions. In this paper the fundamental behavioral irregularities of the vehicle dynamics are identified and suggestions are made on the choice of effectual weighting matrices.

System Description

The dynamic descritpion of the re-entry vehicle will be essentially that given in Ref. 7.

$$\dot{X} = V\cos\gamma\cos\psi \tag{1a}$$

$$\dot{Y} = V\cos\gamma\sin\psi \tag{1b}$$

$$\dot{Z} = V\sin\gamma \tag{1c}$$

$$\dot{\gamma} = -AV^{-1}\cos\phi \tag{1d}$$

$$\dot{\psi} = A(V\cos\gamma)^{-1}\sin\phi \tag{1e}$$

$$\dot{A} = -a(A - A_c) \tag{1f}$$

$$\dot{\phi} = -b(\phi - \phi_c) \tag{1g}$$

where A_c is commanded acceleration and ϕ_c is bank angle command. Equation (1) relates the three position variables (X, Y, Z), the two angular variables (γ, ϕ) and the two autopilot variables (A, ϕ), to the two-dimensional actuating signal (A_c, ϕ_c).

For a given initial condition and actuating signal, Eq. (1) can be integrated and the nominal trajectory deduced. The perturbation equations are then easily derived from an expansion of the right aide of Eq. (1) about the nominal path (see Ref. 1 or 7 for example). Denote the perturbation in the state vector by x and the perturbation in control by u. Then

$$\dot{x} = Fx + Gu \qquad t_o \le t \le t_f \tag{2a}$$

$$x(t_o) = x_0 \tag{2b}$$

where‡

$$F = \begin{bmatrix} 0 & F_{12} \\ 0 & F_{22} \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ a & 0 \\ 0 & b \end{bmatrix}$$

$$F_{12} = \begin{bmatrix} -V\sin\gamma\cos\psi & -V\cos\gamma\sin\psi & 0 & 0 \\ -V\sin\gamma\sin\psi & V\cos\gamma\cos\psi & 0 & 0 \\ V\cos\gamma & 0 & 0 & 0 \end{bmatrix}$$

$$F_{22} = \begin{bmatrix} 0 & 0 & -V^{-1}\cos\phi & V^{-1}A\sin\phi \\ \dfrac{A\sin\phi\sin\gamma}{V\cos^2\gamma} & 0 & \dfrac{\sin\phi}{V\cos\gamma} & \dfrac{A\cos\phi}{V\cos\gamma} \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -b \end{bmatrix}$$

All of the control and trajectory variables are evaluated on the nominal trajectory. The matrix F is time variable on the nominal trajectories of interest, but the accelerations are such that the variation is "slow."

Since x represents the deviation in response from the ideal, u should be selected to make x small in some appropriate sense. In the classical linear-quadratic problem this is made precise by defining a criterion function J as follows

$$J = x'(t_f)P_f x(t_f) + \int_{t_o}^{t_f} (x'(\tau)Q(\tau)x(\tau) + u'(\tau)R(\tau)u(\tau))\,d\tau \tag{4}$$

where $R > 0$, P and $Q \ge 0$ uniformly in t. It is well known that the state feedback control policy which minimized Eq. (4) is given by

$$u = -R^{-1}G'Px \tag{5}$$

where

$$\dot{P} = -F'P - PF + PGR^{-1}G'P - Q \qquad t_o \le t \le t_f \tag{6}$$

$$P(t_f) = P_f$$

If full state feedback is permissible, the control rule given in Eq. (5) has several desirable properties. It is linear and relatively easy to mechanize. The closed-loop system using Eq. (5) is asymptotically stable if certain technical conditions are satisfied. In addition, Eq. (5) is robust in the sense that if Eq. (2) contains additional white noise terms which model turbulence, atmospheric inhomogeneities, unmodeled high-frequency vehicle dynamics, etc., Eq. (5) still would give the control policy which minimizes the expected value of Eq. (4). Further, the closed-loop system possesses attractive sensitivity properties with respect to variations in the elements of (F, G). (see Ref. 8 for example.) Thus, while it is true that Eq. (2) does not explicitly include such important influences as the uncertainties in the aerodynamic coefficients and air density, the deleterious effects of these uncertainties are ameliorated by the guidance law given by Eq. (5).

Although the controller given by Eq. (5) has all of the previously listed attributes for any permissible choice of (P, Q, R), system performance may still be unsatisfactory. For example, though asymptotically stable, the closed-loop sytem may be inadequately damped. Effecting changes in closed-loop damping is accomplished by modifying the weighting matrices in Eq. (4), but unfortunately it is not immediately evident how (P, Q, R) should be changed.

The stability properties of the re-entry vehicle are more easily seen in the transformed state space given by

$$z = Tx \tag{7}$$

$$T = \begin{bmatrix} T_{11} & 0 \\ 0 & I \end{bmatrix}$$

$$T_{11} = \begin{bmatrix} \cos\gamma\cos\psi & \cos\gamma\sin\psi & \sin\gamma \\ -\sin\psi & \cos\psi & 0 \\ -\sin\gamma\cos\psi & -\sin\gamma\sin\psi & \cos\gamma \end{bmatrix}$$

‡To simplify the notation, the dimensions of null matrices and identity matrices will not be given if these dimensions are obvious from the context.

The dynamic equation in the z-coordinate system is

$$\dot{z}=F_z z+Gu \tag{8}$$

where

$$F_z=\begin{bmatrix} F_{z11} & F_{z12} \\ 0 & F_{22} \end{bmatrix}$$

$$F_{z11}=\begin{bmatrix} 0 & \dot{\psi}\cos\gamma & \dot{\gamma} \\ -\dot{\psi}\cos\gamma & 0 & \dot{\psi}\sin\gamma \\ \dot{\gamma} & -\dot{\psi}\sin\gamma & 0 \end{bmatrix} \tag{9}$$

$$F_{z12}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & V\cos\gamma & 0 & 0 \\ V & 0 & 0 & 0 \end{bmatrix}$$

This change in state variable representation resolves the vehicle motion into a component along the velocity vector z_1, an orthogonal component in the plane of motion z_3, and a component perpendicular to the plane of motion z_2.§ Stability and controllability properties are preserved by this nonsingular transformation.

The unforced system given by Eq. (8) and Eq. (9) has anomalous stability properties. On the nominal trajectory of concern here, (F_z, G) are slowly varying. More specifically it will be assumed that A is constant, $\dot{\gamma}$ is small, $\dot{\psi}$ and $\dot{\phi}$ are zero, and $\phi=180°$, $\psi=0°$. If time invariant, the stability of the open-loop system is partially characterized by the open-loop poles which are in turn given by the eigenvalues of F_z. Direct calculation shows that these eigenvalues are located at points given by

$$\{v_i\}=\{0, 0, 0, \pm j(\dot{\psi}^2+\dot{\gamma}^2)^{1/2}, -a, -b\} \tag{10}$$

Only those poles attributable to the autopilot are in the left half plane. The poles of the vehicle are all on the imaginary axis and indeed there are five poles very near the origin. While a system with poles given by Eq. (10) could under special circumstances be stable, only through feedback could it be made asymptotically stable.

Equation (8) is time varying and thus Eq. (10) gives only a qualitative indication of the open-loop stability properties. Whatever its open-loop properties, if Eq. (8) is controllable, the closed-loop system using Eq. (5) as its guidance law is asymptotically stable. A controllable system is one in which any initial error can be eliminated with a linear feedback control law in an arbitrarily short time.¶ Under the assumption that the coefficient matrices are sufficiently smooth, an algebraic criterion for controllability can be deduced.[9] Define the matrix sequence $\{M_i(t)\}$ by

$$M_0(t)=G(t)$$

$$M_{k+1}(t)=-F_z(t)M_k(t)+\dot{M}_k(t) \qquad k=0, 1, \ldots \tag{11}$$

and let

$$C_j(t)=[M_0, M_1, \ldots, M_{j-1}] \tag{12}$$

The system described by Eq. (2) is controllable if rank $C_n(t)=n$.

§If z is a vector, z_i is its ith component. The vector $\bar{z}_i$ is a unit vector in the ith direction.

¶This is actually a strong form of controllability, but this definition will suffice for the system under study here.

The matrix $C_j(t)$ given by Eq. (12) bears a striking resemblance to the controllability matrix $\overline{C_j}$ for a time invariant system

$$\overline{C_j}(t)=[F_z, F_zG, F_z^2G, \ldots] \tag{13}$$

Direct calculation shows that

$$\text{rank}\ C_n(t)\equiv 7 \tag{14a}$$

$$\text{rank}\ \overline{C_n}(t)\equiv 6 \tag{14b}$$

Indeed, the first 7 columns of $C_j(t)$ are linearly independent for all t when the nominal acceleration is constant. The implications of Eq. (14) are important in this application. Equation (14a) is sufficient to guarantee that the closed-loop guidance law displayed in Eq. (5) is asymptotically stable for appropriately restricted weightings in Eq. (4). Equation (14b) indicates that the degree of stability may be inadequate. To see why this is so, observe that the state space of a linear system may always be decomposed into a set of states or modes that is controllable and a set that is controllable. These former modes are always stabilizable by linear feedback while the latter are unaffected by linear feedback. Hence, if the uncontrolled modes are not asympototically stable, neither will be the closed-loop system.

The system in question is controllable [see Eq. (14a)], but for any fixed time (F, G) does not satisfy the conditions for time invariant controllability. The system could be described as being locally uncontrollable with an uncontrollable mode which rotates with time. Detailed analysis shows that the locally uncontrollable direction in state space rotates in a subspace spanned by the eigenvectors of $F(t)$ which have zero damping. The system is then locally unstabilizable in the sense that for every fixed t, the constant matrix $[E(t), G(t)]$ is not stabilizable. If (F, G) were a rapidly varying function of time, no adverse effect would be expected from these local properties. Unforunately, the nominal trajectory considered for this vehicle results in very slowly varying dynamic matrices in Eq. (2). It is to be expected, therefore, that the closed-loop system will exhibit peculiarities normally associated with systems which are not asymptotically stable.

Selection of the (P, Q, R) Matrices

The guidance law, Eq. (5), is parameterized by the weighting in Eq. (4). Care must be exercised in the selection of these weights to insure that performance specifications will be met. Primary concern in this paper will be oriented toward making judicious choice of (P, Q). The commonly used rule-of-thumb form for R will be employed

$$R(t)=\text{diag}([\Delta A_m^2]^{-1}, [\Delta\phi_m^2]^{-1}) \tag{15}$$

where ΔA_m and $\Delta\phi_m$ are the maximum permissible magnitude variations in the actuating signals.

Equations (2) and (8) can be rewritten in terms of a normalized control vector

$$\dot{x}=Fx+GR^{-1/2}(R^{1/2}u) \tag{16a}$$

$$=Fx+\tilde{G}\tilde{u} \tag{16b}$$

$$x(t_o)=x_o \tag{16c}$$

$$\dot{z}=F_z z+\tilde{G}\tilde{u} \tag{16d}$$

$$z(t_o)=Tz_o \tag{16e}$$

Since Eq. (2) is controllable, the matrix $W(t_1, t_2)$ defined by

$$W(t_1, t_2)=\int_{t_1}^{t_2}\Phi_z(t_1, \tau)G(\tau)R^{-1}(\tau)G(\tau)'\Phi_z(t_1, \tau)\mathrm{d}\tau \tag{17}$$

is positive symmetric for $t_2 > t_1$. The matrix Φ_z is the transition matrix associated with F_z. Indeed the control $u^*(t)$ which transfers the state $z(t_1)$ at time t_1 to the origin at time t_2 while minimizing

$$\int_{t_1}^{t_2} u'(\tau)R(\tau)u(\tau)\mathrm{d}\tau$$

is given in Ref. 10

$$u^*(t) = -\bar{G}'(t)\Phi_z'(t, t_1)W(t_1, t_2)^{-1}z(t_1) \tag{18}$$

and

$$E(t_1, t_2) = \int_{t_1}^{t_2} u^*(\tau)R(\tau)u^*(\tau)\mathrm{d}\tau = z(t_1)'W(t_1, t_2)^{-1}z(t_1) \tag{19}$$

The quantity E measures the normalized control "energy" required to make the state transfer.

Suppose that $t_2 - t_1 = \Delta$ and Δ is very small. By expanding the matrix W in a power series using its defining equation, Eq. (17), the structure of Eq. (19) is made more apparent. Define the sequence $\{\bar{M}_k(t)\}$ from Eq. (11) using

$$\bar{M}_0(t) = \bar{G}(t)$$

Let $\mathfrak{N}_j(t)$ be the left null space of $\bar{M}_j$

$$\mathfrak{N}_j(t) = \{z : z \in \mathbb{R}^7, z'\bar{M}_j(t) = 0\}$$

Let $\bar{\mathfrak{N}}_j$ be the complement of $\mathfrak{N}_j$. Expanding Eq. (19) on the nominal trajectory it follows that to first order if $z(t_1)$ is an eigenvector of $W(t_1, t_1 + \Delta)$ and if

$$z(t_1) \subset \sum_{k=0}^{i-1} \mathfrak{N}_k \prod \bar{\mathfrak{N}}_i$$

the error can be eliminated with a control of energy proportional to $\Delta^{-(1+2i)}$ (see the Appendix). The implication of this result is important. For example, if $z(t_o)$ is a linear combination of the columns of $\bar{G}$[or $\bar{M}_0(t_1)$], the error can be eliminated by direct action of $\bar{u}$ and the energy is proportional to Δ^{-1}. Moreover, if z_α and z_β are orthogonal eigenvectors of $W(t_1, t_1 + \Delta)$ and if $z_\alpha'\bar{M}_j(t_1) = [w_1, 0]$, $z_\beta'\bar{M}_j(t_1) = [0, w_2]$, then the energy required to transfer z_α to the origin is, for small Δ, equal to $(w_2/w_1)^2$ times the energy required to cause similar transfer of z_β to the origin. The direction z_β is said to be more controllable than z_α by the factor $|w_2/w_1|$. Clearly if $z(t) \in \mathfrak{N}_j(t)$ for all j, it cannot be transferred to the origin at all.

Direct but tedious calculation of the $\bar{M}_k$ produces the result that (see the Appendix): C1) $\bar{z}_6$ is more controllable than $\bar{z}_7$ by the factor $a\Delta A_m / b\Delta\phi_m = \lambda_1$; C2) $\bar{z}_2$ is more controllable than $\bar{z}_5$ by the factor $a \cos\gamma\Delta A_m / Ab\Delta\phi_m = \lambda_2$; C3) $\bar{z}_2$ is more controllable than $\bar{z}_3$ by the factor $Ab\Delta\phi_m / a\Delta A_m = \lambda_3$; and C4) the energy required to eliminate $\bar{z}_1$ is proportional to $\dot{\gamma}^{-2}$. With these relations in mind, it is possible to make a judicious choice of (P, Q). Since the conditions are stated in the z coordinate system, the weighting matrices (P_z, Q_z) in this coordinate system will first be found.

The states z_6 and z_7 are autopilot errors and, at this time, there are no penalities associated with their variation. Consequently a reasonable choice for their weightings in Q would be

$$(Q_z)_{66} = (Q_z)_{77} = 0 \tag{20}$$

The states z_4 and z_5 are flight-path and azimuth angular error respectively. Suppose these error variables are required to stay within nominal bounds parameterized by q_2; i.e.

$$z_r^2(t) + z_5^2(t) \le 2/q_2^2$$

The usual rule-of-thumb selection for Q_z would satisfy

$$(Q_z)_{44} = (Q_z)_{55} = q_2^2$$

Suppose $\lambda_2 < 1$. Then z_5 is easier to control than z_4 (see C2). For equal initial errors, the residaul error in the $\dot{z}_4$ direction at times greater than t_o will tend to dominate that in $\bar{z}_5$ because of the difficulty in applying control effort to z_4. To cause the closed-loop damping in these two modes to be more nearly the same, a heavier weight in Q_z should be assigned to z_4. To increase the weighting in z_4 while maintaining the same overall angular deviation, let Q_z satisfy

$$(Q_z)_{44} + (Q_z)_{55} \le 2q_2^2 \qquad \frac{(Q_z)_{55}}{(Q_z)_{44}} = \lambda_2^2$$

Solving this equation

$$(Q_z)_{44} = \frac{2}{1+\lambda_2^2} q_2^2 \qquad (Q_z)_{55} = \frac{2\lambda_2^2}{1+\lambda_2^2} q_2^2 \tag{21}$$

The important trajectory variables z_2 and z_3 measure errors orthogonal to the nominal trajectory. Reasoning as above, if

$$z_2^2 + z_3^2 \le 2/q_1^2$$

then

$$(Q_z)_{22} = \frac{2}{1+\lambda_3^2} q_1^2 \qquad (Q_z)_{33} = \frac{2\lambda_3^2}{1+\lambda_3^2} q_1^2 \tag{22}$$

The final state variable measures motion along the nominal velocity vector. As C4 indicates, errors in this direction are quire difficult to control. Errors in z_1 require energy proportional to $\dot{\gamma}^{-2}$ to correct and in this application $\dot{\gamma}$ is small. It is easy to see what gives rise to this conspicuous expenditure of control energy. Neither normal acceleration nor bank angle commands create any first-order change in tangential velocity. If the flight-path angle were constant ($\dot{\gamma} = 0$) errors along the velocity vector could not be eliminated (the system is not stabilizable). Since $\dot{\gamma} \equiv 0$, some control over z_1 is possible. This is accomplished by shortening or lengthening the turning radius of the vehicle. To provide regulation of path length, sizable amounts of control effort are required to produce small variations in z_1. Because of the weak coupling between u and z_1, if z_1 has a weighting in Q_z, u will have a tendency to give exclusive attention to this error.

If the absolute time of evolution along the trajectory is of little concern, the system can be made insensitive to tangential errors by making $(Q_z)_1 = 0$. In this application this is a reasonable choice. The final form for Q_z becomes

$$Q_z = \mathrm{diag}\left(0, \frac{2}{1+\lambda_3^2} q_1^2, \frac{2\lambda_3^2}{1+\lambda_3^2} q_1^2, \frac{2}{1+\lambda_2^2} q_2^2, \frac{2\lambda_2^2}{1+\lambda_2^2} q_2^2, 0, 0\right) \tag{23}$$

At the terminal time, only position impact errors are of concern. An error in z_2 at $t = t_f$ corresponds approximately to an impact error of size $|z_2|$ if the impact time is nearly equal to t_f. An error in z_3 could give rise to an error of size $|z_3[\sin\gamma(t_f)]^{-1}|$. If the circular error is confined by the constraint that it be less than $(2q_3^{-2})^{1/2}$, then

$$P_z = \mathrm{diag}\left(0, \frac{2}{1+\lambda_3^2} q_3^2, \frac{2\lambda_3^2}{(1+\lambda_3^2)\sin^2\gamma(t_f)} q_3^2, 0, 0, 0, 0\right) \tag{24}$$

Equations (15, 23, 24) give weighting matrices in the z-coordinate system. The actual observations and on-line calculations take place in the x-coordinate system. It can be shown directly that a regulator problem in z-coordinate system with respect to weighting (P_z, Q_z, R_z) is equivalent to a regulation problem in the x-coordinate system with respect to $(T' P_z T, T' Q_z T, R_z)$ in the sense that the corresponding regulators are identical.

From Eqs. (7, 23, and 24)

$$P_f=\begin{bmatrix} (P_f)_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$(P_f)_{11}=\frac{2q_3^2}{1+\lambda_3^2}\begin{bmatrix} \lambda_3^2\cos^2\psi+\sin^2\psi & \ldots & \ldots \\ (\lambda_3^2-1)\cos\psi\sin\psi & \lambda_3^2\sin^2\psi+\cos^2\psi & \ldots \\ -\lambda_3^2\cot\gamma\cos\psi & -\lambda_3^2\cot\gamma\sin\psi & \lambda_3^2\cot^2\gamma \end{bmatrix} \tag{25}$$

$$Q=\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}$$

$$Q_{11}=\frac{2q_1^2}{1+\lambda_3^2}\begin{bmatrix} \sin^2\psi+\lambda_3^2\sin^2\gamma\cos^2\psi & \ldots & \ldots \\ (\lambda_3^2-1)\cos\psi\sin\psi & \cos^2\psi+\lambda_3^2\sin^2\gamma\sin^2\psi & \ldots \\ -\lambda_3^2\sin\gamma\cos\gamma\cos\psi & -\lambda_3^2\sin\gamma\cos\gamma\sin\psi & \lambda_3^2\cos^2\gamma \end{bmatrix}$$

$$Q_{22}=\mathrm{diag}\left(\frac{2}{1+\lambda_2^2}q_2^2, \frac{2\lambda_2^2}{1+\lambda_2^2}q_2^2, 0, 0\right) \tag{26}$$

Equations (15, 25, and 26) give the weighting matrices in the x-coordinate system.

Example

The previous section provides a procedure for selecting the weighting matrices to be used in the synthesis of a guidance law for an aerodynamically controlled re-entry vehicle. To illustrate the utility of these weighting matrices a simple example is useful. In this example, two controllers will be compared. The first u_1, uses the performance index shown above while the second, u_2, uses weighting matrices which might have been chosen after viewing a simulation of system performance. This is not to suggest that u_2 is the best choice on this empirical basis, but it does represent a reasonable base of comparison. Denote the weighting matrices associated with the ith controller by (P_i, Q_i, R_i).

The sample trajectory will be a constant acceleration curve in the plane. It will be assumed that $\psi\equiv 0°$, $\phi_c\equiv\phi\equiv 180°$. The quantity R_1 measures the available perturbation acceleration of the vehicle.

This is a complicated function of time on the nominal trajectory. Suffice it to say that: $R_{11}(t_o)$ is of order 10^{-5}; and $R_{11}(t_f)$ is of order 10^{-7} with smooth monotonic variation in between. The bank angle limit, though not germane to this example, was restricted to 10°. Using these specifications, λ_2 and λ_3 and R can be computed.

To complete the specification, suppose that the weighting on position deviation increases quadratically with equal weighting assigned to 6×10^3 ft error at t_o and a 10 ft error at t_f; i.e.

$$q_1=\left(6000-5990\frac{(t-t_o)}{(t_f-t_0)}\right)^{-1} \tag{27}$$

and let

$$q_3=q_1(t_f) \tag{28}$$

The weighting for angular error uses a 1° permissible variation, so $q_2\equiv 57.3$. Substituting into the equations for (P, Q) given earlier, (P_1, Q_1) can be computed with $R_1=R$.

To motivate the selection of (P_2, Q_2) some history of this problem is helpful. Early simulations of vehicle performance made apparent the fact that the system had anomalous stability properties. Initial errors in X for example caused Z variations even when $z(t_o)=0$. This system peculiarity was sometimes described by saying that "X and Z could not both be controlled satisfactorily." (Actually it has been shown that only a linear combination of X and Z is strongly controllable.) For this reason it was suggested that a weighting be assigned to one of these states and no weighting be assigned to the other. At impact the value of γ was large and consequently X was the more important error variable and so the weighting on altitude was set equal to zero. The resulting weightings were

$$Q_2=\mathrm{diag}[q_1^2, q_1^2, 0, q_2^2, q_2^2, 0, 0] \tag{29a}$$

$$P_2=\mathrm{diag}[q_1(t_f)^2, q_1(t_f)^2, 0, 0, 0, 0, 0] \tag{29b}$$

$$R_2=R_1=R \tag{29c}$$

The matrices (P_2, Q_2) attempt to quantify the same performance specifications given by (P_1, Q_1). They differ because the relative controllability of the different modes is not included in Eq. (29). Rather similar system errors are given identical weights in Eq. (29) and the local non-stabilizability attributes of the system are treated on an ad hoc basis.

A simple trajectory in the (X, Z) plane was examined.

$x_1(t_o)=$ order 10^4 $\quad Z(t_o)=$ order 10^5

$x_i(t_o)=0, i\neq 1$ $\quad X(t_o)=$ order 10^5

$t_0=0$ $\quad t_f=$ order 10

Some important qualitative features of the system response are shown in Fig. 1. The trajectory associated with control u_i is denoted by x_{pi}. This figure is not drawn to scale in order that effects attributable to the difference in controllers can be made more apparent.

All of the trajectories begin at the same point but x_{p2} crosses x_n and terminates below it. The trajectories have the properties which would be expected from their respective performance indices. The guidance law u_1 reduces the magnitude of the error and rotates its direction so that it is aligned with the velocity vector. The guidance law u_2 on the other hand rotates the error essentially into elevation.

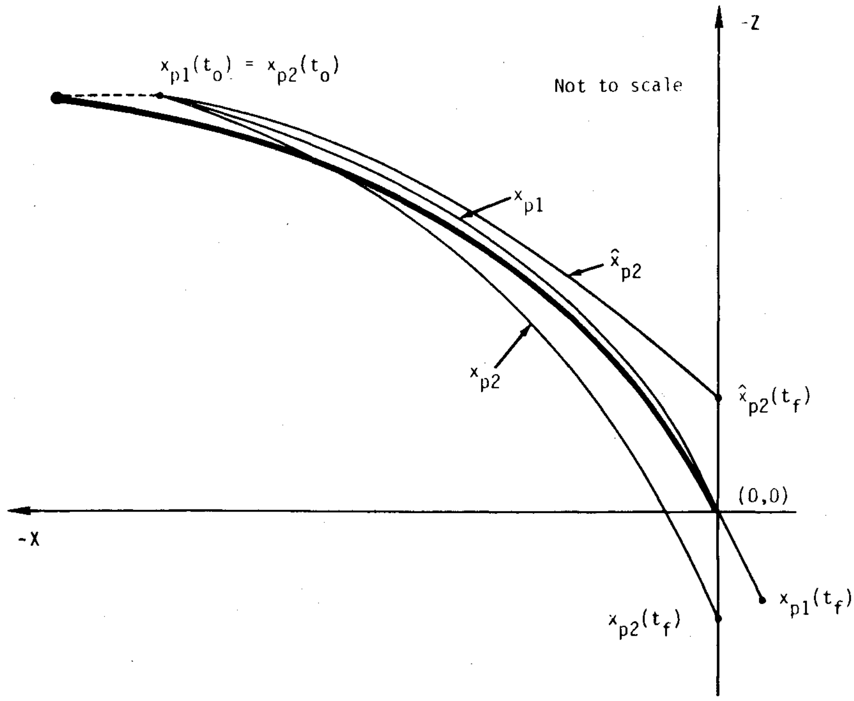

Fig. 1 Comparison of perturbed trajectories.

The character of x_{p2} has one anomaly deserving comment. The matrices (P_2, Q_2) attach no penalty to shifts in elevation and so it is to be expected that x_{p2} would take the form of a motion "parallel" to x_n; i.e., a motion having the same direction but vertically shifted. On this basis another trajectory denoted by $\hat{x}_{p2}$ appears to be a more likely candidate for the trajectory because it is "parallel" and presumably takes less energy to accomplish. The flaw in this line of reasoning is that more than "parallel" motion is required if the penalty accorded by (P_2, Q_2) is to approach zero. "Parallel" motions with time translation give rise to positive penalities because of a perceived error in x. Observe that u_2 must "slow" the system before moving into a "parallel" path. This slowing is accomplished by having x_{p2} cross x_n, and producing an increase in path length thereby. For reasons discussed earlier, this leads to a considerable increase in the energy required from the guidance law. To "slow" the vehicle, the weakly controllable mode must be excited and this necessitates an increase in the size of the actuating signal. For the specific sample trajectory x_{p2} requires roughly twice the energy [see Eq. (19)] required by x_{p1}.

Table 1 gives relative performance of u_1 and u_2 at impact for various initial errors and parameter variations. In every case u_1 provided superior performance. Though it is probably true that u_2 is inferior to other controllers that could have been selected empirically, this example illustrates that the system designer ignores the relative stability of the system modes at his great risk. Only by explicitly modifying the performance penalities can the available guidance energy be allocated in the most appropriate way.

Conclusion

This paper presents an algorithm for synthesizing a guidance law for an aerodynamically controlled re-entry vehicle. This guidance law differs froms its predecessors in that the relative controllability of different system modes enters explicitly into the system penalty function. The point of view espoused here differs from that typically used in that P and Q :re usually constrained to be diagonal and equal weightings are assigned to states that are intended to have equal deviations. It is the fundamental point of this discussion that equal penalties do not cause "equal" responses. Indeed, just the opposite is true. Equal weighting on states will tend to preserve relative responses in closed-loop that the states had in open-loop. Weightings modified by the relative controllability properties of the states will tend to counteract this. Though the problem is conceptually simpler than that studied by Skelton,[4] this procedure is similar in effect to that of Skelton in that the difficult-to-control directions are identified and greater emphasis is placed upon them. The closed-loop controller is expected to display more uniformity in its response to initial errors in different directions than would one not employing the controllability factors.

Table 1 Guidance law comparison

		Impact deviations			
Perturbations	Guidance law	ΔX (ft)	$\Delta\gamma$ (deg)	Δt (sec)	ΔMach (–)
Initial downrange error	u_2	652.	3.8	.11	.21
	u_1	−10.	0.9	.09	.16
Initial altitude error	u_2	−2773.	−6.9	−.36	.09
	u_1	−151.	−2.9	−.33	.19
Density perturbation	u_2	377.	−2.0	.06	−.50
	u_1	−3.	0.1	.05	−.50

Appendix

To deduce conditions C1-C4, note that for any t_o

$$\Phi(t_o),\tau)\tilde{G}(\tau) = \tilde{G}(t_o) + (\tau - t_o)\frac{d}{d\rho}\Phi(t_o,\rho)\tilde{G}(\rho)\Big|_{\rho=t_o}$$

$$+\frac{(\tau-t_o)^2}{2}\frac{d^2}{d\rho^2}\Phi(t_o,\rho)\tilde{G}(\rho)\Big|_{\rho=t_o} + \ldots$$

$$= \sum_{i=0}^{\infty}\frac{(\tau-t_o)^i}{i!}\tilde{M}_i \qquad \text{(A1)}$$

Substituting Eq. (A1) into Eq. (17) and retaining only terms through third order

$$W(t_o,t_1) = \sum_{i,j=0}^{3}\frac{(t_i-t_o)^{i+j+1}}{i!j!(i+j+1)}\tilde{M}_i(t_o)\tilde{M}_j'(t_o) \qquad \text{(A2)}$$

Suppose σ is a normalized eigenvector of $W(t_o,t_1)$ with associated eigenvalue v; i.e.

$$W(t_o,t_1)\sigma = v\sigma \qquad \text{(A3)}$$

Then

$$\sigma' W(t_o,t_1)^{-1}\sigma = v^{-1} \qquad \text{(A4)}$$

From Eq. (19) an error in the direction σ requires at least v^{-1} units of energy if the error is to be eliminated in the period (t_o,t_1).

The eigenvalues of the positive matrix W can be ordered from largest to smallest $(v_7, v_6, \ldots, v_1)$, with corresponding eigenvectors $(\mu_7, \ldots, \mu_1)$. These eigenvectors can be found from the relations[11]

$$v_7 = \max \mu' W\mu = \mu_7' W\mu_7; \text{ subject to } \|\mu\| = 1$$

$$\vdots \qquad \text{(A5)}$$

$$V_1 = \min \mu' W\mu = \mu_1' W\mu_1; \text{ subject to } \|\mu\| = 1$$

Denote $t_1 - t_o$ by Δ. Direct calculation shows $(\bar{z}_6, \bar{z}_7) \in \tilde{\mathfrak{N}}_o$ and

$$\lim_{\Delta\to 0} \mu_7 = \bar{z}_7; \quad \lim_{\Delta\to 0} z_7\Delta^{-1} = b^2\Delta\Phi_m^2 \qquad \text{(A6a)}$$

$$\lim_{\Delta\to 0} \mu_6 = \bar{z}_6; \quad \lim_{\Delta\to 0} z_6\Delta^{-1} = a^2\Delta A_m^2 \qquad \text{(A6b)}$$

For small Δ the ratio of energy required to eliminate errors in $\bar{z}_7$ as compared to $\bar{z}_6$ is $a^2\Delta A_m^2(b^2\Delta\phi_m^2)^{-1}$. Consequently $\bar{z}_7$ will be said to be locally less controllable than $\bar{z}_6$ by the factor $(a\Delta A_m)(b)\Delta\phi_m)^{-1}$.

Direct calculation of M_1 under the assumption that $\mathcal{R}$ is differentiable, leads to the conclusion that $(\bar{z}_4, \bar{z}_5) \in \mathfrak{N}_1\Pi\tilde{\mathfrak{N}}_0$. From Eq. (A5) it follows that

$$\lim_{\Delta\to 0}\mu_5 \quad \text{and} \quad \lim_{\Delta\to 0}\mu_4 \in \mathfrak{N}_1\Pi\mathfrak{N}_2$$

From Eq. (A2) and the fact that $\bar{z}_4$ and $\bar{z}_5$ are eigenvectors of $\tilde{M}_1\tilde{M}_1'$ for all t, it follows that

$$\lim_{\Delta\to\&} \mu_5 = \bar{z}_5; \quad \lim_{\Delta\to 0} v_5\Delta^{-3} = \dot{\gamma}^2 b^2\Delta\phi_m^2(\cos\gamma)^{-2} \qquad \text{(A7a)}$$

$$\lim_{\Delta\to 0} \mu_4 = \bar{z}_4; \quad \lim_{\Delta\to 0} v_4\Delta^{-3} = a^2V^{-2}\Delta A_m^2 \qquad \text{(A7b)}$$

From this it is evident that $\bar{z}_5$ is locally less controllable than $\bar{z}_4$ by the factor $a\Delta A_m\cos\gamma(bA\Delta\phi_m)^{-1}$.

Continuing this procedure, $(\bar{z}_3, \bar{z}_2) \in \tilde{\mathfrak{N}}_2\Pi(\mathfrak{N}_0\Pi\mathfrak{N}_1)$ and have the relative controllability indicated by condition C3. Finally $\bar{z}_1$ spans

$$\prod_0^2 \mathfrak{N}_i$$

for all t. Any error vector $z(t_o)$ can be expanded into components with respect to the natural basis. If $z(t_o)'\bar{z}_k = 0$ for $k > i$ then $z(t_o)$ can be returned to the origin with control of size less than or equal to $\Delta^{-(1+2i)}$.

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